An Introduction to Hypernetworks

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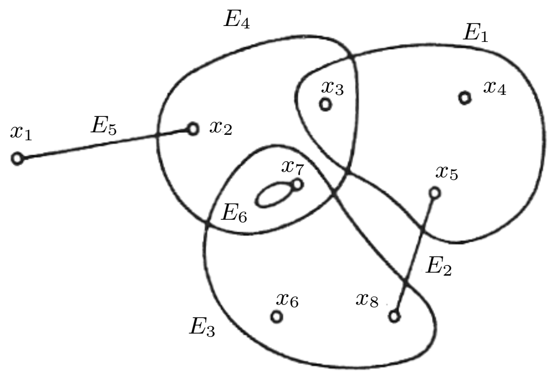
1 Hypergraphs

A *hypergraph* is a set of *vertices*, V, and a set of subsets of V, E called *hypergraph edges*. In general the members of E can have more than two elements.



(a)

$$\begin{array}{cc} & \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ V = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array} & \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right] \end{array}$$

(b)

Figure 1: The Berge hypergraph

The hypergraph shown in Figure 1 is taken from the book *Hypergraphs* by Claude Berge [Berge, 1989]. The vertices are $V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and there are six edges $E = \{E_1, E_2, E_3, E_4, E_5, E_6\}$. The relation, R, between the edges and vertices is given in Figure 1(b). The edges are $E_1 = \{x_3, x_4, x_5\}$, $E_2 = \{x_5, x_8\}$, $E_3 = \{x_6, x_7, x_8\}$, $E_4 = \{x_2, x_3, x_7\}$, $E_5 = \{x_1, x_2\}$ and $E_6 = \{x_7\}$. The set $H_E \stackrel{\text{def}}{=} \{E_1, E_2, E_3, E_4, E_5, E_6\}$ will be called a *hypergraph*.

In Figure 1(a)) the conventional vertex and edge representation of graphs for the loop $E_6 = \{x_7, x_7\}$ and lines $E_2 = \{x_5, x_8\}$ and $E_5 = \{x_1, x_2\}$ is mixed with the Euler circle method of representing sets used for E_1, E_3, and E_4. It would be more consistent to draw the singleton and two-element sets as Euler circles as shown in Figure 2, and this is how hypergraphs will be drawn here.

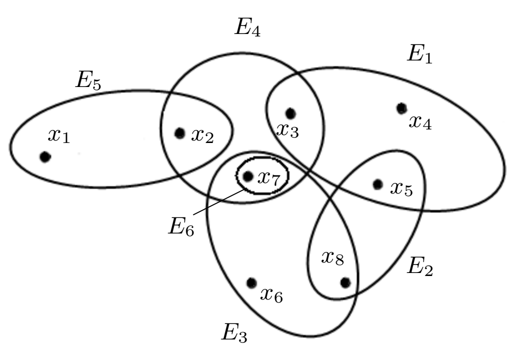
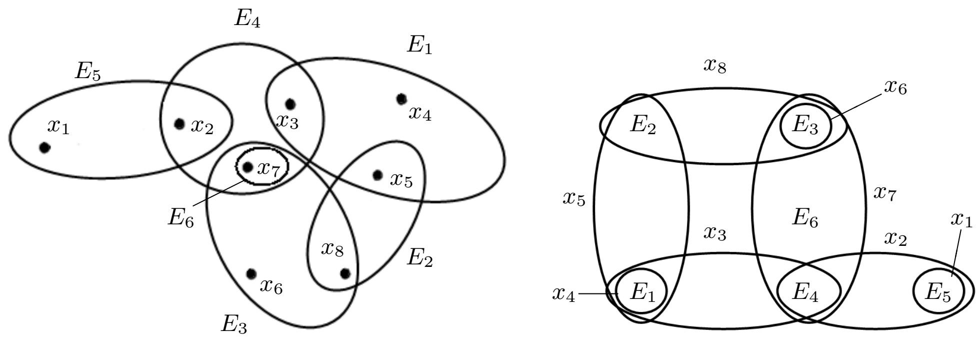


Figure 2: The Berge hypergraph drawn in Euler set form



(a) $H_E(V, R)$ (a) $H_V(E, R^{-1})$

Figure 3: The Dual Berge hypergraphs

Every hypergraph has a *dual* hypergraph as illustrated in Figure 3(b) for the Berge hypergraph. Here the "edges" are sets of edges associated with the vertices, *e.g.* x_2 is associated with the dual edge $\{E_4, E_5\}$.

Let R be a relation between A and B. Let the notation $a\ R\ b$ mean that a is R-related to b. Let $R(a) = \{b \mid \text{for all } b \text{ in } B \text{ with } a\,R\,b\}$ and $R(b) = \{a \mid \text{for all } a \text{ in } A \text{ with } a\,R\,b\}$. In general a relation R between sets A and B has two associated hypergraphs, $H_A(B; R)$ and $H_B(A; R)$, defined as follows:

$$H_A(B; R) \stackrel{\text{def}}{=} \{R(a) \mid a \text{ in } A),$$

$$H_B(A; R) \stackrel{\text{def}}{=} \{R(b) \mid b \text{ in } B).$$

As an example let $A = \{$London Bus, London Taxi, Postbox$\}$ and $B = \{$red, black, big, small, wheels, slot, metal$\}$. The relation R and the hypergraph $H_A(B; R)$ are shown in Figure 4. The dual hypergraph is $H_B(A; R)$ with members R(big) $= \{$London Bus$\}$, R(black) $= \{$London Taxi$\}$, R(slot) $= \{$London Postbox$\}$, R(wheels) $= \{$London Bus, London Taxi$\}$, R(red) $= \{$London Bus, London Postbox$\}$, R(small) $= \{$London Taxi, London Postbox$\}$, and R(metal) $= \{$London Bus, London Taxi, London Postbox$\}$.

$H_A(B; R)$

R(London Bus) $= \{$big, red, metal, wheels$\}$

R(London Taxi) $= \{$small, black, metal, wheels$\}$

R(Postbox) $= \{$small, red, metal, slot$\}$

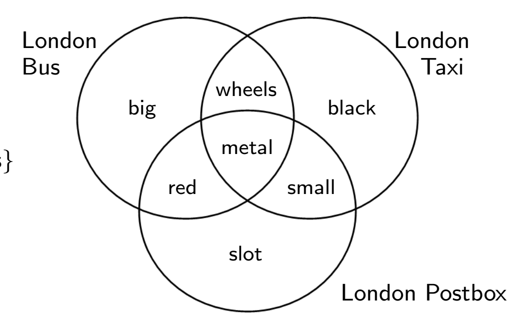


Figure 4: The Hypergraph $H_A(B; R)$

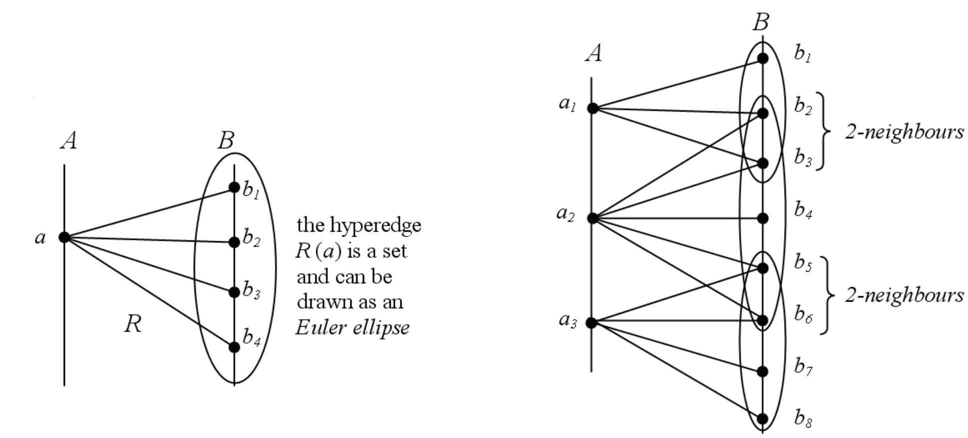
2 The Hypergraphs of a Bipartite Network

A relation R between sets A and B has an associated network with vertices $A \cup B$ and edges (a, b) where $a\, R\, b$. Bipartite relations with $A \cap B = \emptyset$ have a much richer connectivity structure than might appear at first sight. As before let $R(a) = \{\, b \,|\ b \in B \text{ with } \ a\, R\, b\}$, *e.g.* $R(a) = \{b_1, b_2, b_3, b_4\}$ in Figure 5(a). Then

$$
\begin{aligned}
H_A(B; R) &= \{R(a_1),\ R(a_2),\ R(a_3)\} \\
&= \{\, \{b_1, b_2, b_3\},\ \{b_2, b_3, b_4, b_5, b_6\}, \{b_5, b_6, b_7, b_8\}\,\}
\end{aligned}
$$

$$
\begin{aligned}
H_B(A; R) &= \{R(b_1),\ R(b_2),\ R(b_3),\ R(b_4),\ R(b_5),\ R(b_6),\ R(b_7),\ R(b_8)\} \\
&= \{\, \{a_1\},\ \{a_1, a_2\},\ \{a_2\},\ \{a_2, a_3\},\ \{a_3\},\ \}
\end{aligned}
$$

are the hypergraph edges of R in Figure 5(b). The sets $R(a)$ and $R(b)$ are called *hyperedges*. The number of vertices in a hyperedge $R(a)$ is called its *extent*, written $|R(a)|$.



(a) the hyper-edge $R(a)$ (b) connected hyper-edges

Figure 5: Connected hyperedges in a bipartite network

Let the hyperedge $R(a)$ be a *neighbour* of the hyperedge $R(a')$ if their intersection is non-empty, $R(a) \cap R(a') \neq \emptyset$. $R(a)$ is an *h-neighbour* of $R(a')$ if $|R(a) \cap R(a')| \geq h$.

Hyperedges $R(a)$ and $R(a')$ are said to be *h-connected* under R if there exists a sequence $a_1, a_2, ..., a_\ell$ with $a = a_1$, $a' = a_\ell$, with $R(a_i)$ being an h-neighbour of $R(a_{i+1})$ for $i = 1, ..., \ell - 1$.

Figure 5(b) illustrates this. $R(a_1)$ is a 2-neighbour of $R(a_2)$ and $R(a_2)$ is a 2-neighbour of $R(a_3)$. Thus a_1 and a_3 are h-connected for $h = 2$.

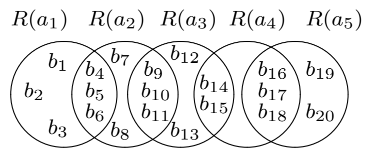


Figure 6: a_1 is 2-connected to a_5

More generally Figure 6 shows a *chain* of h-connected hyperedges where $h = 2$ is the smallest value of $|R(a_i) \cap R(a_{i+1})|$.

Being h-neighbours is an important property in networks. Generally $R(a) \cap R(a')$ provides structure for a_1 to interact with a_2 and the set of all pairwise intersections can play an important role in the dynamics of systems. To establish notation let $R(\{a, a'\}) \stackrel{\text{def}}{=} R(a) \cap R(a')$. Although pairwise intersections are clearly important, why stop there? For example, why not consider $R(\{a, a', a''\}) \stackrel{\text{def}}{=} R(a) \cap R(a') \cap R(a'')$?

3 The Galois Connection

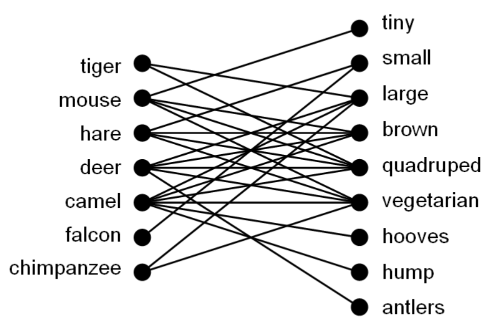


Figure 7: An animal – characteristic bipartite network

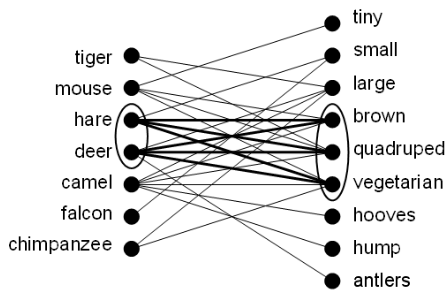
Let R be a relation between A and B, and let A' be a subset of A, $A' \subseteq A$. Let

$$R(A') \stackrel{\text{def}}{=} \bigcap_{a \in A'} R(a).$$

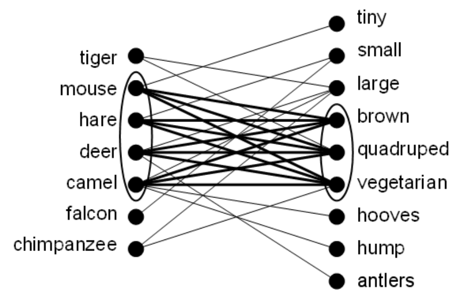
This definition allows the intersection of the $R(a)$ to be formed from any subset of A. For example, Figure 7 shows a relation between a set of animals, A, and a set of their features, F. Let $A' = \{$ mouse, hare, deer, camel $\}$, where

R(mouse) = { tiny, brown, quadruped, vegetarian },
R(hare) = {small, brown, quadruped, vegetarian },
R(deer) = {large, brown, quadruped, vegetarian, hooves, antlers}, and
R(camel) = {large, brown, quadruped, vegetarian, hooves, hump}.

Then $R(A') = \bigcap_{a \in A'} R(a) = \{$ brown, quadruped, vegetarian $\}$.



(a) $R(\{\text{hare, deer}\}) =$
$\{\text{brown, quadruped, vegetarian}\}$



(b) $R(R(\{\text{hare, deer}\}) =$
$\{\text{mouse, hare, deer, camel}\}$

Figure 8: $\{\text{hare, deer}\} \subset R^2(\{\text{hare, deer}\}) = \{\text{mouse, hare, deer, camel}\}$

Of course for many subsets A' of A the intersection $\bigcap_{a \in A'} R(a)$ will be empty. For example, in Figure 8 $R(\{\mathsf{tiger, mouse, chimpanzee}\}) = \{\mathsf{large, quadruped}\} \cap \{\mathsf{tiny, brown, quadruped, vegetarian}\} \cap \{\mathsf{small, vegetarian}\} = \emptyset$.

Although R is a relation between sets A and B, by an abuse of notation the same symbol is used to define a mapping from the power set of A (set of all subsets) to the power set of B, $R : \mathcal{P}(A) \to \mathcal{P}(B)$ with $R : A' \to R(A')$ for all A' in $\mathcal{P}(A)$. In general R is many-one, *e.g.* $R(\{\mathsf{tiger, mouse}\}) = \{\mathsf{quadruped}\} = R(\{\mathsf{tiger, hare}\})$.

By another abuse of notation let R also represent a mapping from the power set of B to the power set of A, $R : \mathcal{P}(B) \to \mathcal{P}(A)$ with $R : B' \to R(B')$. In the literature if R is a relation between A and B, the relation "going the other way" from B to A is sometimes written as R^{-1}, so that $a\,R\,b$ if and only if $b\,R^{-1}\,a$. However, our abuse of notation makes the development simpler. In particular, the symbol R^2 can be used for the double application of R, $\mathcal{P}(A) \overset{R}{\to} \mathcal{P}(B) \overset{R}{\to} \mathcal{P}(A)$ to give $R^2 : \mathcal{P}(A) \to \mathcal{P}(A)$.

As illustrated in Figure 8 $A' \subseteq R^2(A')$ for all $A' \subseteq A$ (assuming that A has no isolated vertices). $A' \subseteq A$ is defined to be *maximal* under R if $A' = R^2(A')$, and $B' \subseteq B$ is *maximal* under R if $B' = R^2(B')$. Then

If A' is a maximal subset of A then $R(A')$ is a maximal subset of B.
If B' is a maximal subset of B then $R(B')$ is a maximal subset of A.

To see this, let $R(A') = B'$. If A' is maximal then $A' = R^2(A') = R(R(A')) = R(B')$. Then $R(A') = R(R(B'))$ so $B' = R^2(B')$ and B' is maximal. A similar argument shows $R(B')$ is maximal.

The hypergraph $\mathcal{H}_A(B; R) \overset{\text{def}}{=} \{R(A') \,|\, \text{ for all maximal } A' \subseteq A\ \}$ will be called the *Galois* hypergraph of $H_A(B; R)$. The hypergraph $\mathcal{H}_B(A, R) \overset{\text{def}}{=} \{R(B') \,|\, \text{ for all maximal } B' \subseteq B\ \}$ will be called the *Galois* hypergraph of $H_B(A; R)$.

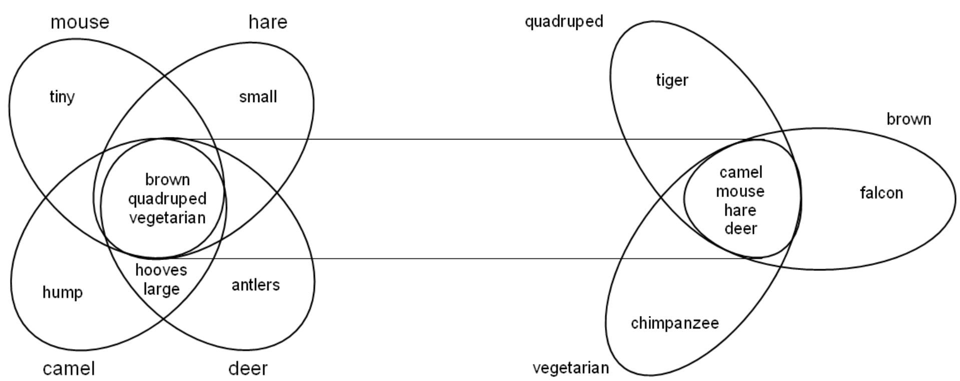


Figure 9: Maximal sets are paired in the Galois Hypergraphs of a relation

The mappings $R : \mathcal{H}_A(B;R) \to \mathcal{H}_B(A;R)$ and $R : \mathcal{H}_A(B;R) \to \mathcal{H}_B(A;R)$ are one-to one. Together they form what is called a *Galois Connection*.

This is illustrated in Figure 9 for the animal-characteristic relation in Figure 8. On the left are the hyperedges $R(\text{camel}) \cap R(\text{mouse}) \cap R(\text{hare}) \cap R(\text{deer}) = \{\text{brown, quadruped, vegetarian}\}$ in $\mathcal{H}_A(B,R)$. On the right right are the hyperedges $R(\text{brown}) \cap R(\text{quadruped}) \cap R(\text{vegetarian}) = \{\text{camel, mouse, hare, deer}\}$ in $\mathcal{H}_A(B,R)$. The Galois connection establishes the *Galois pair* relationship $A' \leftrightarrow B'$ where $R(A') = B'$ and $R(B') = A'$, for example

$$\{\text{brown, quadruped, vegetarian}\} \leftrightarrow \{\text{camel, mouse, hare, deer}\}$$

The Galois connection is considered by many to be a particularly beautiful structure. Among many elegant properties it has the following:

For all maximal A' and A'' either $R(A') \cap R(A'') = \emptyset$ or $A' \cup A''$ and $A' \cap A''$ are maximal. For all maximal B' and B'' either $R(B') \cap R(B'') = \emptyset$ or $B' \cup B''$ and $B' \cap B''$ are maximal. Furthermore

$$R(A' \cup A'') = R(A') \cap R(A''), \qquad R(A' \cap A'') = R(A') \cup R(A'').$$

$$R(B' \cup B'') = R(B') \cap R(B''), \qquad R(B' \cap B'') = R(B') \cup R(B'').$$

4 Galois Pairs and Maximal Rectangles

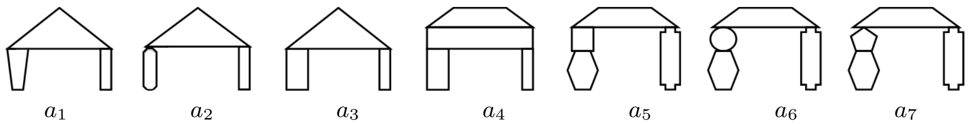


Figure 10: Arches related to the blocks used to construct them

Figure 10 shows a set of arches, $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ with each arch made from a subset of the blocks $B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}\}$. Let a be R-related to b if it contains block b. This bipartite relation can be represented by an incidence matrix as shown in Figure 11. The entry in the i^{th} row and the j^{th} column of the matrix is one if a_i is related to b_j, and it zero otherwise.

In a Galois pair $A' \leftrightarrow B'$ every a in A' is R-related to every b in B'. Therefore the rows and columns of the matrix can be rearranged so that all the a_i in A' are contiguous and all the b_j in B' are contiguous, with the corresponding rectangle of entries in the matrix all ones. For example, let $A' = \{a_1, a_2, a_3\}$ and $B' = \{b_3, b_4\}$. Then as shown in Figure 11 the corresponding rectangle is filled with ones because each of a_1, a_2 and a_3 is related to b_3 and b_4.

The rectangle corresponding to $A' = \{a_1, a_2, a_3\} \leftrightarrow B' = \{b_3, b_4\}$ is *maximal*. Two other maximal rectangles are shown in Figure 11 corresponding to the Galois pairs $\{a_3, a_4\} \leftrightarrow \{b_4, b_5\}$ and $\{a_5, a_6, a_7\} \leftrightarrow \{b_7, b_8, b_9\}$. The maximal rectangles $A' \leftrightarrow B'$ where A' has just one element or B' has just one element are not shown.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
a_1	1	0	1	1	0	0	0	0	0	0	0	0
a_2	0	1	1	1	0	0	0	0	0	0	0	0
a_3	0	0	1	1	1	0	0	0	0	0	0	0
a_4	0	0	0	1	1	1	1	0	0	0	0	0
a_5	0	0	0	0	0	0	1	1	1	1	0	0
a_6	0	0	0	0	0	0	1	1	1	0	1	0
a_7	0	0	0	0	0	0	1	1	1	0	0	1

Figure 11: Maximal rectangles in the arch-block structure

5 The Galois Lattice

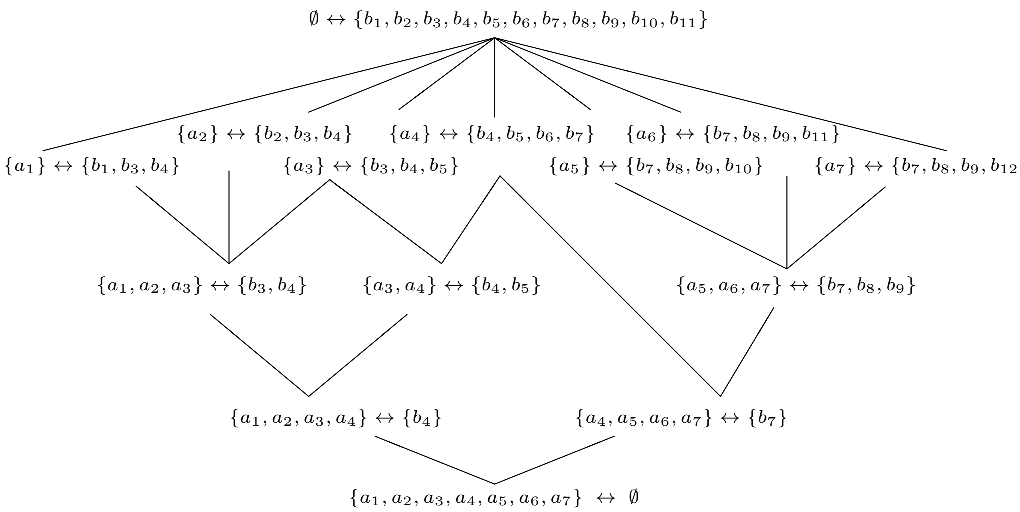


Figure 12: The Galois Lattice for the arch-block relation of Figure 11

The Galois pairs form a partially ordered set induced by set ordering. Let $A' \leftrightarrow B'$ and $A'' \leftrightarrow B''$ be Galois pairs. Then $A' \subset A''$ if and only if $B' \supset B''$. Thus the Galois pairs can be arranged as a lattice, also called a *Hasse diagram* or a *construct lattice*. The Galois lattice for the arch-block structure is shown in Figure 12. Figure 13 gives another example. In this case the *supremum* of the lattice is the Galois pair $\{1, 2, 3, 4, 5\} \leftrightarrow \emptyset$ and the *infimum* is $\emptyset \leftrightarrow \{a, b, c, d, e, f, g, h, i\}$.

R	a	b	c	d	e	f	g	h	i
1	1	0	1	0	0	1	0	1	0
2	1	0	1	0	0	0	1	0	1
3	1	0	0	1	0	0	1	0	1
4	0	1	1	0	0	1	0	1	0
5	0	1	0	0	1	0	1	0	0

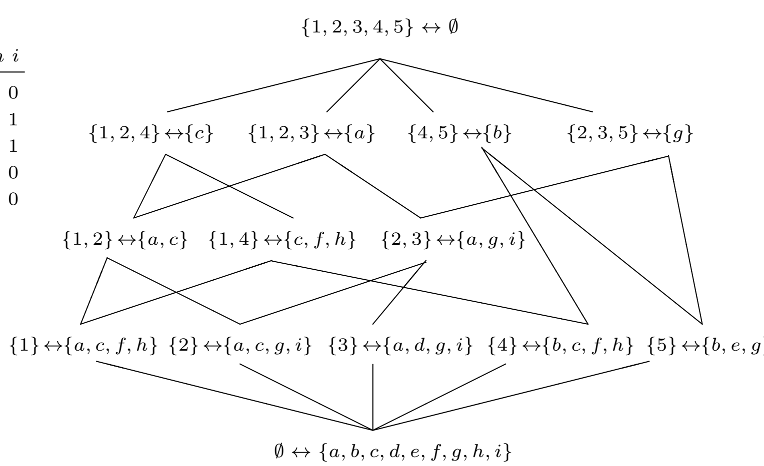
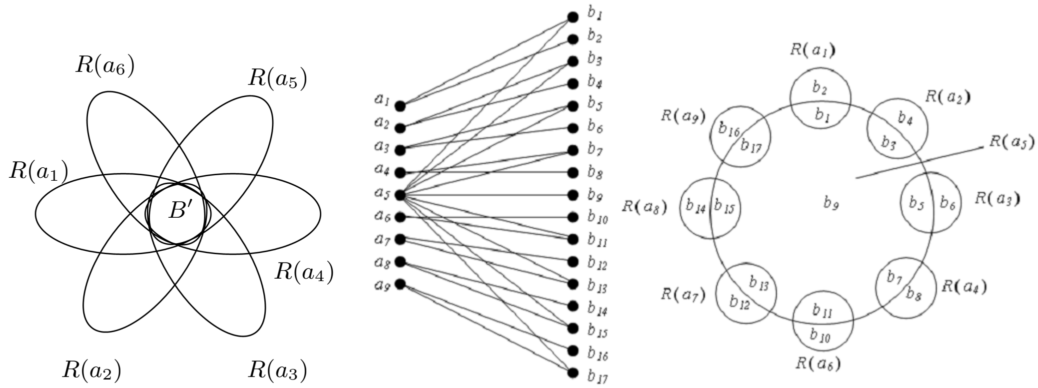


Figure 13: A relation R and its Galois Lattice

6 Weak and Strong Connectivity in Hypergraphs



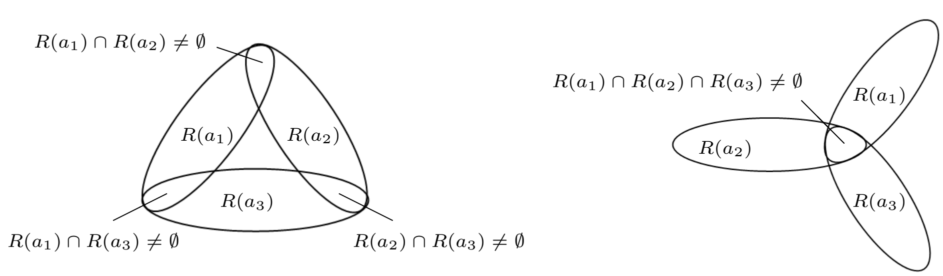
(a) strong connectivity (b) weakly connected neighbours of a_5

Figure 14: Strong and weak connectivity

Galois pairs are sites of connectivity and potential interaction in hypergraphs. For the pair $A' \leftrightarrow B'$ with $A' = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ shown in Figure 14(a) the larger the set B' the more highly connected are the elements of A'. Let the *hub* of a set of hypergraph edges be their intersection, $\text{hub}(A_1, A_2, ..., A_n) \stackrel{\text{def}}{=} \cap_{i=1}^{n} A_i$.

Let the *neighbourhood* of a in A be the set $\mathcal{N}_A(a) \stackrel{\text{def}}{=} \{a' \mid R(a) \cap R(a') \neq \emptyset\}$. Figure 14(b) shows an extreme case in which none of the members of the neighbourhood $\mathcal{N}_A(a_5)$ intersects any of the others, apart from $R(a_5)$, so that $\text{hub}(\mathcal{N}_A(a)) = \emptyset$. $\mathcal{N}_A(a_5)$ is said to have *weak* connectivity.

Figure 15 illustrates a fundamental difference in the way hypergraph hyperedges can be configured. In Figure 15(a) the hyperedges intersect each other pairwise, but their hub is empty. In this case the configuration is not as highly connected as the configuration in Figure 15(b). When the hub of a neighbourhood is non-empty, the neighbourhood will be said to be *strongly connected*.



(a) $\mathcal{N}(a_1) \cap \mathcal{N}(a_2) \cap \mathcal{N}(a_3) = \{a_1, a_2, a_3\}$
$R(a_1) \cap R(a_2) \cap R(a_3) = \emptyset$

(b) $\mathcal{N}(a_1) \cap \mathcal{N}(a_2) \cap \mathcal{N}(a_3) = \{a_1, a_2, a_3\}$
$R(a_1) \cap R(a_2) \cap R(a_3) \neq \emptyset$

Figure 15: Strongly and weakly connected neighbourhoods

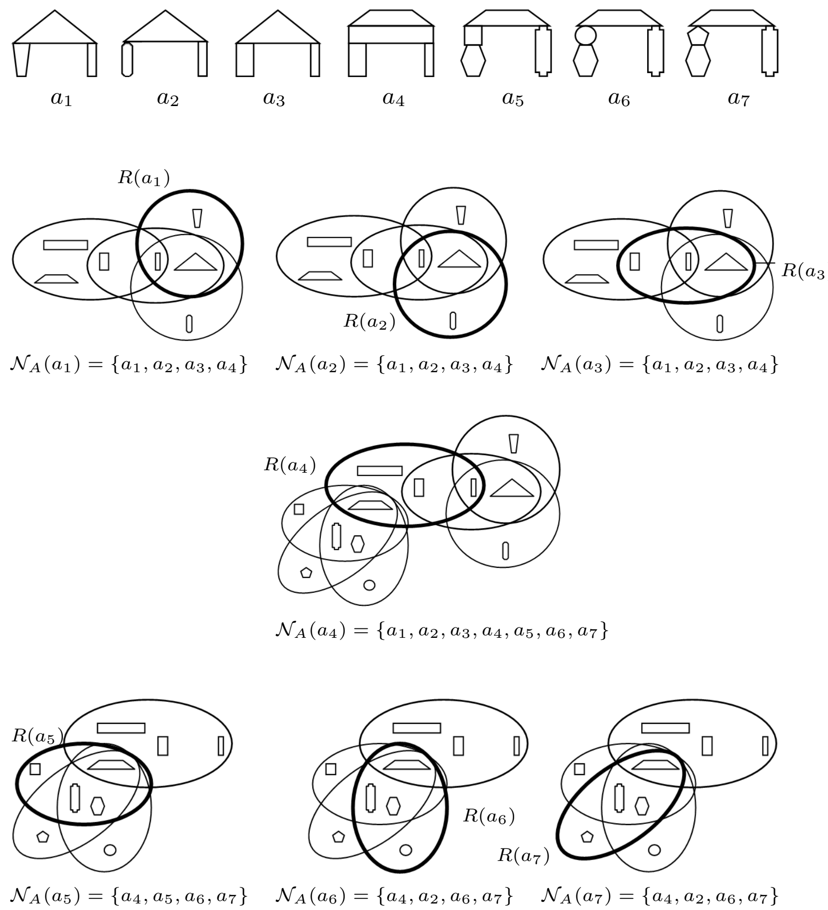


Figure 16: Neighbourhoods in the arch-block structure

Figure 16 illustrates these definitions for the arch-block structures. The neighbourhoods for a_1, a_2 and a_3 are all the same, as are those for a_5, a_6, a_7. The neighbourhood of a_4 contains all the other other arches. As can be seen, the hyperedge $R(a_4)$ bridges the cluster of hyperedges $R(a_1)$, $R(a_2)$, and $R(a_3)$ with the cluster of hyperedges $R(a_5)$, and $R(a_6)$ and $R(a_7$. Thus $R(a_4)$ connects the hypergraph.

References

[Berge, 1989] Berge, C., *Hypergraphs*, Elsevier Science Publishers B.V. (Amsterdam), 1989.

[Johnson,2014] Johnson, J. H., *Hypernetworks in the science of complex systems*, Imperial College Press (London), 2014.